Physical Meaning of Hermiticity and Shortcomings of the Composite (Hermitian + non-Hermitian) Quantum Theory of Günther and Samsonov

Ali Mostafazadeh

Department of Mathematics, Koç University,
34450 Sariyer, Istanbul, Turkey
amostafazadeh@ku.edu.tr

Abstract

In arXiv:0709.0483 Günther and Samsonov outline a "generalization" of quantum mechanics that involves simultaneous consideration of Hermitian and non-Hermitian operators and promises to be "capable to produce effects beyond those of standard Hermitian quantum mechanics." We give a simple physical interpretation of Hermiticity and discuss in detail the shortcomings of the above-mentioned composite quantum theory. In particular, we show that the corresponding "generalization of measurement theory" suffers from a dynamical inconsistency and that it is by no means adequate to replace the standard measurement theory.

PACS number: 03.65.Ca, 11.30.Er, 03.65.Pm, 11.80.Cr

In [1] we proved the following theorem.

Theorem: *The lower bound on the travel time (upper bound on the speed) of unitary evolutions is a universal quantity independent of whether the evolution is generated by a Hermitian or non-Hermitian Hamiltonian.*

A direct implication of this theorem, which contradicts the main result of [2], is that *as far as the Brachistochrone problem is concerned the use of non-Hermitian (in particular $\mathcal{PT}$-symmetric) Hamiltonians that are capable of generating unitary time-evolutions does not offer any advantage over the Hermitian Hamiltonians.* This is in complete agreement with the earlier results on the physical equivalence of the pseudo-Hermitian (in particular $\mathcal{PT}$-symmetric) quantum mechanics and the standard (Hermitian) quantum mechanics, [3, 4]. To avoid this equivalence, Günther and Samsonov [5] have recently outlined a composite quantum theory involving both Hermitian and

quasi-Hermitian operators [6]. They state that this theory is a genuine generalization of the standard (Hermitian) quantum mechanics in the sense that it allows for "quantum mechanical setups which are capable to produce effects beyond those of standard Hermitian quantum mechanics" and avoids the consequences of the above-stated theorem of [1]. The main purpose of the present paper is to offer a closer look into this composite quantum theory and reveal its shortcomings. In particular, we will show that this theory is by no means an alternative to the standard quantum mechanics or its pseudo-Hermitian representation.

To make our treatment precise, we first list the postulates of pseudo-Hermitian quantum mechanics and discuss their implications.

(A1) A quantum system S is determined by a triplet $(\mathcal{H}, H, \prec\cdot,\cdot\succ)$, where $\mathcal{H}$ is a (separable) Hilbert space with defining inner product $\langle\cdot|\cdot\rangle$, $H : \mathcal{H} \to \mathcal{H}$ is (a densely defined closed) linear operator, called the Hamiltonian, and $\prec\cdot,\cdot\succ$ is a (positive-definite) inner product on $\mathcal{H}$ that may be different from $\langle\cdot|\cdot\rangle$.

(A2) The (pure) states of S are identified with the rays Λ (one-dimensional subspaces) of $\mathcal{H}$. They may be represented by any nonzero element ψ of Λ.[1]

(A3) The observables of the system are identified with a class of (densely defined closed) linear operators $O_\alpha : \mathcal{H} \to \mathcal{H}$. Upon measuring an observable O_α while the system is in a state Λ one obtains a reading $\omega \in \mathbb{R}$ and the state Λ that is prepared before the measurement undergoes an abrupt change into an eigenstate Λ' of O_α with eigenvalue ω. Alternatively, any state vector $\psi \in \Lambda$ collapses to an eigenvector $\psi_\omega \in \Lambda'$ of O_α.

(A4) All the eigenvalues (and eigenvectors) of an observable O_α are among the possible outcomes of a measurement of O_α. In particular, every eigenvector of O_α may be prepared in this way.

(A5) The outcome of a measurement is probabilistic in nature and the probability $P_\omega(\Lambda)$ of measuring a non-degenerate[2] eigenvalue ω upon measuring O_α in a state Λ is given by

$$P_\omega(\Lambda) = \frac{|\prec\psi,\psi_\omega\succ|^2}{\sqrt{\prec\psi,\psi\succ\prec\psi_\omega\psi_\omega\succ}}, \tag{1}$$

where $\psi \in \Lambda$ and $\psi_\omega \in \Lambda_\omega$. Furthermore, the expectation (mean) value of the readings obtained by measuring O_α in a state Λ has the form

$$\langle O_\alpha\rangle_\Lambda = \frac{\prec\psi, O_\alpha\psi\succ}{\prec\psi,\psi\succ}. \tag{2}$$

(A6) The time-evolution of an initial state Λ_0 is governed by the Schrödinger equation,

$$i\hbar\frac{d}{dt}\,\psi(t) = H\psi(t), \qquad t \geq t_0 \tag{3}$$

[1] As the association of states Λ to state vectors ψ is one to infinitely many, we make a distinction between states and state vectors.

[2] The generalization to non-degenerate eigenvalues is straightforward. We do not consider it for brevity.

subject to the initial condition

$$\psi(t_0) = \psi_0, \tag{4}$$

where ψ_0 is any state vector belonging to the initial state Λ_0. The evolving state $\Lambda(t)$ is uniquely determined by the condition $\psi(t) \in \Lambda(t)$.

(A7) The Hamiltonian H is an observable.

Note that the postulates (A1) – (A7) do not involve the requirement that the observables or the Hamiltonian be Hermitian or that the time-evolution be unitary. They have however a number of drastic consequences. Some of the most important of these are the following.

(B1) There is a dense set of state vectors that can be prepared for any physical measurement involving an observable O_α. This follows from (2) and the assumption that observables are densely defined, i.e., (A3).

(B2) The observables must have a real spectrum. This follows from (A4).

(B3) The observables must have a complete set of eigenvectors. As discussed in [7], this follows from (A3) – (A5) and (B1).

(B4) The observables O_α must be Hermitian with respect to the inner product $\prec\cdot,\cdot\succ$. This follows from a well-known theorem [8] of linear algebra saying that the reality of the expectation values of O_α in every state is equivalent to the Hermiticity of O_α with respect to $\prec\cdot,\cdot\succ$.

(B5) The time evolution (operator $e^{-i(t-t_0)H/\hbar}$) is unitary with respect to the inner product $\prec\cdot,\cdot\succ$. This follows from (A7) and (B4).

Note that the theorem mentioned in (B4) provides a clear physical meaning for the requirement of the Hermiticity of observables, namely that

Hermiticity of an observable means reality of its expectation values.[3]

This seems to be overlooked by those claiming that Hermiticity is a mathematical requirement that lacks physical meaning and hence must be replaced by other "physical" conditions [9].

The postulates (A1) – (A7) of pseudo-Hermitian quantum mechanics reduce to those of conventional (Hermitian) quantum mechanics, if we demand that $\prec\cdot,\cdot\succ$ and $\langle\cdot|\cdot\rangle$ are identical. The advantage of pseudo-Hermitian quantum mechanics is that it includes the choice of $\prec\cdot,\cdot\succ$ as a degree of freedom.

In pseudo-Hermitian quantum mechanics the physical Hilbert space $\mathcal{H}_{\rm phys}$ of a quantum system is obtained by endowing the span of the eigenvectors of H with the inner product $\prec\cdot,\cdot\succ$ and Cauchy completing the resulting inner product space into a Hilbert space. Therefore, $\mathcal{H}_{\rm phys}$ is determined once one makes a choice for $\prec\cdot,\cdot\succ$. The arbitrariness in the choice of $\prec\cdot,\cdot\succ$ does not however lead to a genuine generalization of Hermitian quantum mechanics. It turns out that

[3]This is a more stringent condition than the reality of the spectrum.

there is a unitary transformation $\rho : \mathcal{H}_{\rm phys} \to \mathcal{H}$ mapping the observables $O_\alpha : \mathcal{H}_{\rm phys} \to \mathcal{H}_{\rm phys}$ onto the Hermitian operators $o : \mathcal{H} \to \mathcal{H}$ in such way that the probabilities of measurements and expectation values are left invariant, [4]. This means that a given physical system may be represented equally well using Hermitian or pseudo-Hermitian quantum mechanics and that there is no physical quantity capable of distinguishing between them; they are not different theories but different representations of the same theory.

The results of [2] seem to challenge the physical equivalence of Hermitian and pseudo-Hermitian quantum mechanics, for they imply that using a class of pseudo-Hermitian Hamiltonians one can achieve arbitrarily fast unitary time evolutions (which is known to be impossible in Hermitian quantum mechanics.) The above-mentioned theorem of [1] contradicts this claim openly [1]. In [5] the authors attempt at devising a more general framework than the one given by (A1) – (A7) so that they could provide an alternative interpretation for the results of [2] and avoid the no-go theorem of [1]. We will show that this attempt is unsatisfactory.

Before discussing the results of [5], we wish to point out that there is nothing wrong with the calculations of [2]. If one defines the Hilbert space using an inner product that renders the time evolution unitary[4], then the calculations of [2] amount to establishing the obvious fact that the travel time between arbitrarily close initial and final states is arbitrarily small. However, if one relaxes the requirement of the unitarity of time evolution, one can indeed achieve arbitrarily fast evolutions between distant states. The latter is a plausible approach in an effective treatment of open systems.

It should also be emphasized that the possibility of non-unitary arbitrarily fast evolutions is not surprising. A simple example of a non-unitary evolution that achieves arbitrarily short travel times for distant states is the following. Consider the standard two-level system where the Hilbert space is $\mathbb{C}^2$ endowed with the Euclidean inner product, and let the Hamiltonian of the system be given by

$$H := \begin{pmatrix} E & a \\ 0 & E \end{pmatrix}. \tag{5}$$

where $E, a \in \mathbb{R}$ are constants. Then the state $\Lambda_0 = \Lambda(0)$ represented by the normalized state vector $\psi_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ evolves into the state $\Lambda(t)$ represented by a normalized state vector of the form

$$\psi(t) := e^{i\varphi(t)} \begin{pmatrix} i\left(1 + \frac{a^2t^2}{\hbar^2}\right)^{-1/2} \\ \left(1 + \frac{\hbar^2}{a^2t^2}\right)^{-1/2} \end{pmatrix}, \tag{6}$$

where $\varphi(t)$ is an arbitrary real-valued function of t. Now, suppose that t is any positive real number and that $|a|$ is so large that $\epsilon := \frac{\hbar}{|a|t} \ll 1$. Then

$$\psi(t) = e^{i\varphi(t)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{O}(\epsilon) \tag{7}$$

[4] This is what is done in [2].

In other words we can perform a spin flip in arbitrarily short time $t > 0$ provided that $|a| \gg \frac{\hbar}{t}$. The main difference between this model and the one considered in [2] is that the Hamiltonian of the latter is Hermitian with respect to a non-Euclidean inner product. But if one defines the Hilbert space using this new inner product, then $S_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is no longer an observable and its eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ do not represent spin states. Indeed in the limit that the travel time (for the corresponding unitary evolution) tends to zero, the states defined by these state vectors coincide; there is indeed no evolution!

In [5] the authors adopt a view consisting essentially of a superposition (simultaneous consideration) of the Hermitian and pseudo-Hermitian representations of quantum mechanics together with what they refer to as a "generalization" of the measurement theory. The latter involves defining an observable as a conjugacy class of a Hermitian operator. This scheme leaves many of the physically relevant questions about theory unanswered. The most important of these is the process of measurement itself. Suppose that one makes a measurement of an observable in a state and reads ω. According to [5] the observable is a conjugacy class of a Hermitian operator. But this information is not sufficient to interpret the outcome of the measurement. Though one might still identify ω with an eigenvalue of all the operators in the conjugacy class (as they are isospectral), the state obtained after the measurement cannot be an eigenstate of all these operators. With no additional assumption on the faith of the state after the measurement, we do not have a viable physical theory with any sort of predictive power.

If we set up the rule, as suggested in [5], that one must use the Hermitian element of the conjugacy class and apply the standard measurement theory to this operator, then one must also deal with the issue of the uniqueness of this Hermitian element. In principle, a conjugacy class may include more than one Hermitian operator. A more serious problem is that if the process of measurement is only sensitive to Hermitian operators, then what is the reason for considering their conjugacy classes as observables? A possible answer would be to note that the dynamics is generated by a non-Hermitian Hamiltonian H. This latter assumption leads to a host of even more severe difficulties, particularly when one considers evolving observables, i.e., using the Heisenberg picture. For example, the Hamiltonian H will not evolve in time, but there will be other operators in the conjugacy class of H such as the Hermitian Hamiltonian h that will change in time. Clearly, as a set of operators the conjugacy class of H will depend on time. Should now one say that the energy is conserved? Given that one has to perform energy measurements using h, the energy conservation will be lost!

Next, consider the outcome of an energy measurement. Following the prescription of [5], i.e., performing the measurement using h, the state will collapse into an eigenstate of h. If one defines the dynamics using H, because h and H do not generally commute, repeated measurements of energy will yield totally different values irrespective of how short the time interval between two successive measurements is. This is simply unacceptable, for it destroys the predictive power of the theory. If, on the other hand, one defines the dynamics using h, then one has nothing but the

conventional quantum mechanics, because the measurement is done using Hermitian operators and the dynamics is generated by a Hermitian Hamiltonian. The presence of non-Hermitian operators in the scheme do not affect the physical aspects of the problem. Hence they can be safely discarded.

In summary, the so-called "composite system" scheme of [5] that is offered as a way of avoiding the above-mentioned no-go theorem of [1] suffers from very basic consistency problems. Our attempt to resolve these problems leads to a reduction of this composite quantum theory to the conventional quantum mechanics (equivalently to its pseudo-Hermitian representation). This is because a consistent treatment of a physical system that undergoes both time-evolution and measurement requires the use of a Hamiltonian and an observable that are both Hermitian with respect to the same positive-definite inner product. Using observables that are Hermitian together with a Hamiltonian that is non-Hermitian leads to a dynamical inconsistency reminiscent of the inconsistency [4, 10] arising from the identification of observables in $\mathcal{PT}$-symmetric quantum mechanics with the $\mathcal{CPT}$-symmetric operators with real spectrum [9].

References

[1] A. Mostafazadeh, arXiv:0706.3844, to appear in Phys. Rev. Lett.

[2] C. M. Bender, D. C. Brody, H. F. Jones, and B. K. Meister, Phys. Rev. Lett. **98**, 040403 (2007), arXiv:quant-ph/0609032.

[3] A. Mostafazadeh, J. Phys. A **36**, 7081 (2003), arXiv:quant-ph/0304080;
A. Mostafazadeh and A. Batal, J. Phys. A **37**, 11645 (2004), arXiv:quant-ph/0408132.

[4] A. Mostafazadeh, Czech J. Phys. **54**, 1125 (2004), arXiv:quant-ph/0407213.

[5] U. Günther and B. Somsonov, arXiv:0709.0483v1.

[6] F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne, Ann. Phys. (NY) **213** 74 (1992).

[7] A. Mostafazadeh, Czech J. Phys. **56**, 919 (2006), arXiv:quant-ph/0606173.

[8] S. Axler, *Linear Algebra Done Right,* Springer, New York, 1997.

[9] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002), arXiv:quant-ph/0208076; ibid, Am. J. Phys. **71**, 1095 (2003), arXiv:hep-th/0303005.

[10] A. Mostafazadeh, arXiv:quant-ph/0407070.